We have seen that $M_0(\Gamma) = \mathbb{C}$,

but if we allow poles on $\Gamma \backslash \hat{H}$ then we have a very important modular function.

<u>Defn</u> The <u>Klein's j-invariant</u> is defined as the function

$$j(z) := \frac{E_4^3(z)}{\Delta}, \quad \Delta = \frac{E_4^3 - E_6^2}{1728}.$$

$$= 1728 \frac{E_4^3(z)}{E_4^3(z) - E_6^2(z)}$$

$$= \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n$$

$c_1 = 196884$, $c_2 = 2^{11} \cdot 5 \cdot 2099 = 21443760$.
$2^3 \cdot 3^3 \, 1823$

The dim of smallest non-trivial irreducible repn of the sporadic simple group of largest size, the so called Monster group is $196883 = c_1 - 1$

The other coefs $c_n$ also relate to the dimensions of reps of the Monster.

This was first observed by John McKay which is the starting point of the Moonshine conjecture. (The term was coined by John Conway and Simon Norton in 1979 who made several conjectures. They are proved by Richard Borcherds in 1992.

The Moonshine theory is beyond the scope of this course but we can prove some other properties of the $j$-function.
It is easy to see that $j(\gamma z) = j(z)$ $\forall \gamma \in \Gamma$.
We have the following mapping property of $j$.

<u>Thm 3-10</u> The function $j: \mathbb{H} \to \mathbb{C}$ induces a bijection from $_{\Gamma}\backslash\overline{\mathbb{H}}$ (sides identified and $\infty$ included) and the Riemann sphere

$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

<u>Idea of proof</u>: Since $j(\gamma z) = j(z)$ $\forall \gamma \in \Gamma$, $j(z)$ depends only on the orbit of $z$ for any $z \in \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.
Hence it induces a well defined map on $\Gamma\backslash\overline{\mathbb{H}}$

Since $\Delta$ is a cusp form $\Delta(\infty) = 0$ but and $E_4^3(\infty) \neq 0$. Hence $j(\infty) = \infty$

Hence $j: {}_{\Gamma}\backslash\overline{\mathbb{H}} \longrightarrow \mathbb{C} \cup \{\infty\}$

Since $\infty$ is the only zero of $\Delta(z)$
$j(z) \neq \infty$ if $z \neq \infty$

To show that the map $j: {}_{\Gamma}\backslash\overline{\mathbb{H}} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a bijection

It is enough to consider $w \in \mathbb{C}$.

We need to show that $\exists!\ z \in {}_{\Gamma}\backslash\overline{\mathbb{H}}$ s.t

$j(z) = w$.

Consider the function $f_w(z) := j(z) - w$

We want to show that $f_w(z)$ vanishes on ${}_{\Gamma}\backslash\overline{\mathbb{H}}$ at exactly one point $P \in {}_{\Gamma}\backslash\overline{\mathbb{H}}$

$$f_w(z) = j(z) - w = \frac{E_4^3(z) - w\Delta(z)}{\Delta(z)}$$

Since $\Delta(z)$ vanishes only at $\infty$ of order 1.
and $E_4^3(z) - w\Delta(z)$ does not vanish at $\infty$ (because $E_4(\infty) \neq 0$), if $f_w(z)$ vanishes at some point $z$, it must a finite point in $\Gamma\backslash\mathbb{H}$, and it must be a zero of the numerator $E_4^3(z) - w\Delta(z)$
The function $E_4^3 - w\Delta$ is holom at $\infty$ and is of wt 12, and $n_\infty(E_4^3 - w\Delta) = 0$ (since it does not have a zero or pole at $\infty$)
But then the valence formula applied to $E_4^3 - w\Delta \in M_{12}(\Gamma)$ gives

$$\frac{1}{2} n_i + \frac{1}{3} n_\rho + \sum_{p \neq i, \rho} n_p = \frac{k}{12} = 1$$

The only solns of this eqn in positive integers are 1) $n_i = 2$, $E_4^3 - w\Delta$ vanish at $i$ w/ order 2
or 2) $n_\rho = 3$ $E_4^3 - w\Delta$ " " $\rho$ w/ order 3
or 3) $n_p = 1$ at exactly 1 point $p \in \Gamma\backslash\mathbb{H}$, $p \neq i, \rho$
and order of vanishing is 1.
In any case $E_4^3 - w\Delta$ vanish at exactly one point on $\Gamma\backslash\mathbb{H}$ $\blacksquare$

Note the proof of Thm 3.10
in particular says that

Except for 0 and $12^3 = 1728$

$j$ takes every complex value exactly once.

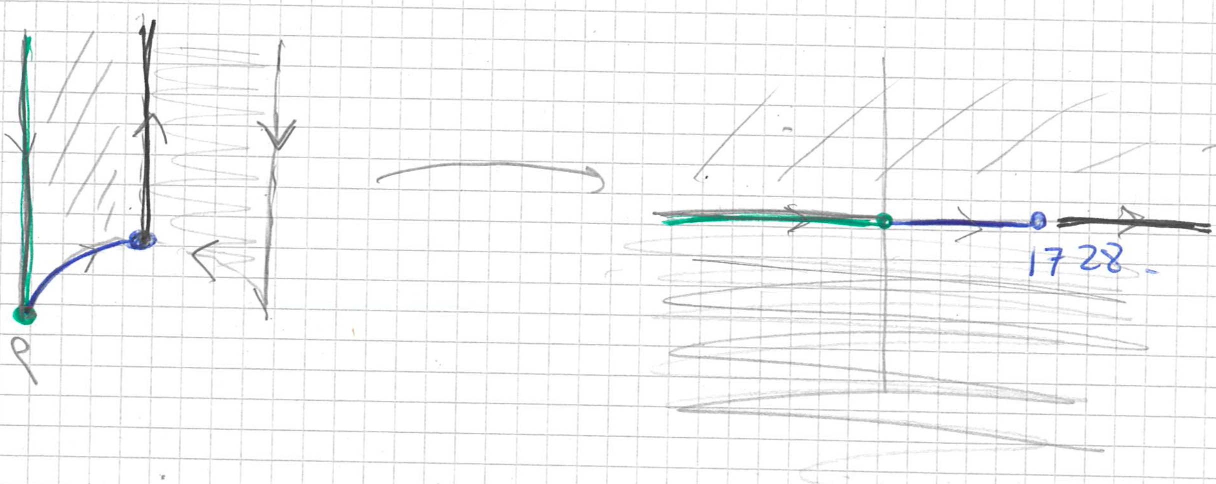
Since $j(z) = \frac{E_4^3(z)}{\Delta} = \frac{E_4^3(z) - E_6^2(z) + E_6^2(z)}{\Delta}$

$= 1728 + \frac{E_6^2(z)}{\Delta}$

Since $E_4(\rho) = 0$ and $E_6(i) = 0$
$j(\rho) = 0$ and $j(i) = 1728$.
$\rho$ is a zero of $j$ of order 3 and at $i$
$j$ takes the value 1728 and $j - 1728$ has
a zero of order 2 at $i$.

One can determine the mapping behavior
of $j$ even more explicitly.

(See Koecher-Krieg, sec. III.5.3
for details
or Apostol sec 2.7)




Next we show that every meromorphic modular function of wt 0 is a rational function in j.

Prop 3-11: Meromorphic modular functions of wt 0 are precisely the rational functions in j.

Proof. Clearly a rational function in j is a modular function of wt 0.

Conversely suppose $f(z)$ is modular of wt 0. If $z_1, \dots, z_t$ are the poles of $f$ in $\Gamma \backslash \mathbb{H}$ counted with multiplicity then

$$f(z) \prod_{l=1}^{t} \left( j(z) - j(z_l) \right)$$ is a modular function of wt 0 with no poles in $\mathbb{H}$. wlog assume $f$ has no poles in $\mathbb{H}$.

We can multiply by a suitable power of $\Delta$ to also cancel the pole at $\infty$.

Then $\Delta^k(z) f(z) \in \mathcal{M}_{12k}(\Gamma)$

Hence $\qquad \Delta^k f_j = \sum_{4i+6j=12k} c_{ij} E_4^{\,i} E_6^{\,j}$

Hence

$$f = \sum_{4i+6j=12k} c_{ij} \frac{E_4^{\,i} E_6^{\,j}}{\Delta^k}$$

So it is enough to show that each $\dfrac{E_4^{\,i} E_6^{\,j}}{\Delta^k}$ with $4i+6j=12k$ is a rational function in $j$

Since $12 \mid 4i+6j$, $\quad 6 \mid 2i+3j$,

$i = 3i_0$, $j = 2j_0$ for some $i_0, j_0$

$$\frac{E_4^{3i_0} E_6^{2j_0}}{\Delta^{i_0+j_0}} = \left(\frac{E_4^{\,3}}{\Delta}\right)^{i_0} \left(\frac{E_6^{\,2}}{\Delta}\right)^{j_0}$$

this is $(j(z))^{i_0}$

As for $\left(\dfrac{E_6^{\,2}}{\Delta}\right)^{i_0}$ recall $\quad \Delta = \dfrac{E_4^{\,3} - E_6^{\,2}}{1728}$

Hence $\quad \dfrac{E_6^{\,2}}{\Delta} = \dfrac{1728\,\Delta - E_4^{\,3}}{\Delta} = 1728 - \dfrac{E_4^{\,3}}{\Delta}$

$$= 1728 - j$$

Hence $\left(\dfrac{E_6^{\,2}}{\Delta}\right)^{i_0} = (1728 - j)^{i_0}$ and we're done

$\square$

As a corollary of last Prop 3.11 we have

**Cor** For $k \in \mathbb{Z}$, let $V_{2k}$ be the space of meromorphic modular functions of wt $2k$, Then

$$V_{2k} = \left(j'(z)\right)^k \mathbb{C}(j)$$

**Proof.** If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $j(\gamma z) = j(z)$

Hence $$\left(j(\gamma z)\right)' = j'(\gamma z) \cdot \left(\frac{az+b}{cz+d}\right)' = j'(z)$$

$$\Rightarrow \quad j'(\gamma z) \cdot (cz+d)^{-2} = j'(z) \quad \text{Hence}$$

$j' \in V_2$

Hence if $f \in V_{2k}$ then $\dfrac{f}{(j')^k} \in V_0 = \mathbb{C}(j)$

by Prop 3.11 and the corollary follows. $\square$

**Ex**: $$E_4 = \frac{1}{(2\pi i)^2} \frac{(j')^2}{j(j-1728)} \qquad E_6 = -\frac{1}{(2\pi i)^3} \frac{(j')^3}{j^2(j-1728)}$$

$$\Delta = \frac{1}{(2\pi i)^6} \frac{(j')^6}{j^4(j-1728)^3}$$

Pf exercise ?

Rmk ① Prop 3.11 amounts to the well known fact that the only meromorphic functions on $\mathbb{C} \cup \{\infty\}$ are the rational functions.

② If a modular function $f$ of wt 0 has only poles at $\infty$. Then we can in fact from the above proof see that it is a polynomial in $j$. The space $\mathbb{C}[j]$ is denoted by $M_0^!(\Gamma)$ and is called weakly holomorphic modular forms of wt 0.

In general $M_k^!(\Gamma)$ = space of meromorphic modular functions of wt $k$ whose only poles at are $\infty$.

Such an $f$ has F.E $f = \sum_{n \geq n_0} a_n q^n$ for some $n_0 \in \mathbb{Z}$.

To close this section we give the following result for the discriminant function.

<u>Thm 3.12</u> $\Delta = \dfrac{E_4^3 - E_6^2}{1728}$ has the $\infty$-product expansion

$$\Delta = q \prod_{n\geq 1} (1-q^n)^{24}$$

<u>Proof</u> (a) Show that $\frac{d}{dz} \log\left(q \prod (1-q^n)^{24}\right) = 2\pi i E_2(z)$

(b) Use transformation $z^{-2} E_2\left(-\frac{1}{z}\right) = E_2(z) - \frac{6i}{\pi z}$

to show $\frac{d}{dz} \log\left(\frac{\tilde{\Delta}(-1/z)}{z^{12}\tilde{\Delta}(z)}\right) = 0$

where $\tilde{\Delta}(z) = q \prod_{n\geq 1}(1-q^n)^{24}$

(c) Use (b) to show that $\tilde{\Delta}(z) \in S_{12}(\Gamma)$

(d) Use $\dim S_{12}(\Gamma) = 1$ to conclude

The Fourier coefficients of $\Delta$ are called Ramanujan $\tau$ function

$$\Delta = q\prod(1-q^n)^{24} = \sum_{n=0}^{\infty} \tau(n) q^n$$

Ramanujan studied them

numerically and observed that

$$(*)\quad \begin{cases} \tau(nm) = \tau(n)\tau(m) & \text{if } (n,m)=1 \\ \tau(p^{n+1}) = \tau(p)\,\tau(p^n) - p^{11}\tau(p^{n-1}) \end{cases}$$

Soon after Mordell proved them

But it was Hecke who explained these identities by defining certain operators, $T_n$ which we now call Hecke operators, and showing that $\Delta$ is an eigenfunction of $T_n$ $\forall n \in \mathbb{N}$.

In terms of the Dirichlet series associated to $\Delta$, $L(\Delta, s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$, we will see that the identities $(*)$ are equivalent to

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1}$$

<u>Open question</u> DH Lehmer conjecture: $\tau(n) \neq 0$ $\forall n \geq 1$.

It is know for at least $n \leq 10^{15}$

Ramanujan also conjectured that $|\tau(p)| < 2p^{11/2}$ $\forall p$ prime

This conjecture is much deeper than the above multiplicative properties of $\tau(n)$, and is proved by Deligne in 1974 as a consequence of his proof of Weil's conjectures.

Note that $11/2 = \frac{12-1}{2}$.

<u>General Ramanujan-Petersson conjecture</u> for holom. modular forms is that

if $f(z) = \sum a_n q^n \in S_k(\Gamma)$ then $a_n = O(n^{\frac{k-1}{2}})$,

ie $\frac{|a_n|}{n^{\frac{k-1}{2}}}$ remains bounded as $n \to \infty$.

We will prove a weaker bound for $a_n$ which is called <u>Hecke's bound</u> or <u>trivial</u> bound, which brings us to our new section.